SOME $\aleph_0$-BOUNDED SUBSETS OF STONE-ČECH COMPACTIFICATIONS

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ABSTRACT
We characterize the maximal m-bounded extension of an arbitrary completely regular Hausdorff space X. The other principal results are: *Theorem.* Let X be a locally compact, σ-compact non-compact space with no more than $2^{\aleph_0}$ zero-sets. Then assuming the continuum hypothesis, $\beta X - X$ can be written as the union of $2^{2^{\aleph_0}}$ pairwise disjoint, dense $\aleph_0$-bounded subspaces. *Theorem.* Let X be a locally compact, σ-compact metric space without isolated points. Then both the set of remote points of βX and the complement of this set in $\beta X - X$ are $\aleph_0$-bounded.

Introduction

All spaces are assumed to be completely regular Hausdorff spaces. Throughout this paper we shall employ the notation and terminology of the Gillman-Jerison text [3]. In particular, the symbol βX denotes the Stone-Čech compactification of the space X. The symbol [CH] preceding the statement of a theorem will indicate that the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$) is used in the proof of the theorem. The cardinal $2^{\aleph_0}$ will be denoted by the letter c. The cardinality of a set S will be denoted by $|S|$.

Let m be any cardinal number. In [4], the authors define a space X to be *m-bounded* if every subset of X of cardinality no greater than m is contained in a compact subset of X. In §1 we show that every space X has a maximal m-bounded extension, and give a characterization of this extension. The other principal results in this paper are as follows:

2.7 THEOREM [CH]. *Let X be a locally compact, σ-compact, non-compact space with no more than c zero-sets. Then $\beta X - X$ can be written as the union of 2^c disjoint, dense $\aleph_0$-bounded subspaces.*

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3.5 THEOREM. *Let X be a locally compact, σ-compact metric space without isolated points. Then both the set of remote points of βX and the complement of this set in $\beta X - X$ are $\aleph_0$-bounded spaces.*

1. The maximal m-extension of a space

The following result of van der Slot and Herrlich appears as a corollary to Theorem 1 of [5].

1.1 THEOREM. *Let $\mathscr{P}$ be a topological property with the following properties:*

(a) *If each member of a family $\mathfrak{J}$ of spaces has $\mathscr{P}$, then the product space $\Pi_{F \in \mathfrak{J}} F$ has $\mathscr{P}$.*

(b) *If X has $\mathscr{P}$ and S is a closed subspace of X, then S has $\mathscr{P}$.*

(c) *Compact spaces have $\mathscr{P}$.*

Then for each space X there exists a unique "maximal $\mathscr{P}$-extension" of X, denoted by γX, with the following properties

(1) *γX contains a dense copy of X*

(2) *γX has $\mathscr{P}$*

(3) *If Y has $\mathscr{P}$ and if $f: X \to Y$ is continuous, then f can be continuously extended to a function $f^\gamma: \gamma X \to Y$.*

1.2 COROLLARY. *Every space X has a maximal m-bounded extension mX.*

PROOF. Lemma 4 of [4] says that m-boundedness has property (a) of the above theorem, and it is obvious that m-boundedness has properties (b) and (c) also.

1.3 THEOREM. *The space mX can be identified with the set S of all points of βX that belong to the βX-closure of some subset of X of cardinality no greater than m (m is assumed to be infinite).*

PROOF. Let T be a subset of S of cardinality no greater than m. For each $p \in T$ there exists a subset $A(p)$ of X such that $|A(p)| \leqq m$ and $p \in \mathrm{cl}_{\beta X} A(p)$. Thus

$$T \subseteq \bigcup_{p \in T} cl_{\beta X} A(p) \subseteq cl_{\beta X} \bigcup_{p \in T} A(p)$$

But $|\bigcup_{p \in T} A(p)| \leqq m \cdot m = m$ and so $\mathrm{cl}_{\beta X}[\bigcup_{p \in T} A(p)] \subseteq S$. Hence $\mathrm{cl}_S T$ is compact and so S is m-bounded.

Let Y be m-bounded, and let $f: X \to Y$ be continuous. Thus f maps X into βY and so there exists a continuous map $f^\beta: \beta X \to \beta Y$ whose restriction to X is f ([3], 6.5I). Let $f^\beta | S = f^\gamma$. To prove that property (3) of Theorem 1.1 holds, it suffices to show that $f^\beta[S] \subseteq Y$. Let D be a subset of X of cardinality no greater than m, and let $p \in \mathrm{cl}_{\beta X} D$. Now $f^\beta[\mathrm{cl}_{\beta X} D] \subseteq \mathrm{cl}_{\beta Y} f[D]$ as f^β is continuous. However, $|f[D]| \leqq m$ and so $\mathrm{cl}_Y f[D]$ is compact. Thus $\mathrm{cl}_{\beta Y} f[D] = \mathrm{cl}_Y f[D] \subseteq Y$, and so $f^\gamma(p) \in Y$. Thus $f^\gamma[S] \subseteq Y$, and the theorem follows.

2. **A decomposition of** $\beta X - X$

The principal result in this section is Theorem 2.7, which is stated in the introduction.

Recall that a point y of the space Y is called a P-point of Y if every G_δ-set of Y that contains y is a neighborhood of y. A discussion of P-points can be found in [3]. It has been proved in Theorem 4.2 of [8] that, assuming the continuum hypothesis, $\beta N - N$ contains a dense set of 2^c P-points (N denotes the countable discrete space). The following result is an immediate consequence of this and of problems 9.D.1 and 9.M.2 of [3].

2.1 THEOREM [CH]. *Let X be locally compact, realcompact, and non-compact. Then every non-empty open subset of $\beta X - X$ contains 2^c P-points of $\beta X - X$.*

2.2 NOTATION. Let us denote the family of zero-sets of X by $\mathscr{Z}(X)$. If A is a closed subset of the space X, we shall denote the set $(\mathrm{cl}_{\beta X} A) - X$ by the symbol A^*. In particular, $X^* = \beta X - X$. Finally, $P(X)$ will denote the set of P-points of the space X.

Recall that a space X is called an F-space if every cozero-set of X is C^*-embedded in X. F-spaces are discussed in chapter 14 of [3].

2.3 LEMMA. *Let Y be a compact F-space, and let D be a set of P-points of Y. Let S be the set of all points of Y that are in the Y-closure of some countable subset of D. Then both S and $Y - S$ are $\aleph_0$-bounded.*

PROOF. The proof that S is $\aleph_0$-bounded is identical to the proof used in 1.3 to show that mX is m-bounded.

Suppose that $Y - S$ is not $\aleph_0$-bounded. Then there exists a countable subset A of $Y - S$ such that $S \cap \mathrm{cl}_Y A \neq \emptyset$. It follows from the definition of S that there exists a countable subset E of D such that $\mathrm{cl}_Y E \cap \mathrm{cl}_Y A \neq \emptyset$. Since $\mathrm{cl}_Y E \subseteq S$ we have $A \cap \mathrm{cl}_Y E = \emptyset$. As each point of E is a P-point of Y, no point of E is the limit point of a countable subset of Y, and so $E \cap \mathrm{cl}_Y A = \emptyset$. Thus E and A are disjoint closed subsets of the countable space $E \cup A$, and hence are disjoint zero-sets of $E \cup A$. But $E \cup A$ is C^*-embedded in the compact F-space Y by problem 14.N.5 of [3]; hence by theorem 6.9(a) of [3] it follows that $\mathrm{cl}_Y(E \cup A) = \beta(E \cup A)$. Theorem 6.5 IV of [3] says that disjoint zero-sets of the space X have disjoint closures in βX; thus $\mathrm{cl}_Y E \cap \mathrm{cl}_Y A = \emptyset$, which is a contradiction. Thus $Y - S$ is $\aleph_0$-bounded.

2.4 Definition. Let $\mathscr{W}$ be a non-empty family of subsets of a space Y. Then $\mathscr{W}$ is called a P-family of Y it is satisfies the following two conditions:

(i) Each member of $\mathscr{W}$ is a dense subset of Y consisting of $\aleph_1$ P-points of Y.

(ii) Distinct members of $\mathscr{W}$ are disjoint.

2.5 Theorem [CH]. *Let X be locally compact, realcompact and noncompact, and let $|\mathscr{Z}(X)| = c$. Then there exists a P-family of X^* whose cardinality is 2^c.*

Proof. Since $|\mathscr{Z}(X)| = c$, the family $\mathfrak{B} = \{X^* - Z^* : Z \in \mathscr{Z}(X)\}$ has cardinality c. It follows from theorem 6.5(b) of [3] that $\mathfrak{B}$ forms a base for the open subsets of X^*, and hence by 2.1 we can find a dense subset D of $\aleph_1$ P-points of X^*. Hence there is at least one P-family of X^* (namely $\{D\}$).

Let $\mathscr{S}$ be the collection of all P-families of X^*, and order $\mathscr{S}$ by inclusion. The partially ordered set $\mathscr{S}$ contains a maximal chain $\mathscr{C}$, and it is easy to see that $\mathfrak{F} = \bigcup_{\mathscr{W} \in \mathscr{C}} \mathscr{W}$ is a P-family of X^*. We claim that $|\mathfrak{F}| = 2^c$.

Suppose that $|\mathfrak{F}| < 2^c$. As each set belonging to $\mathfrak{F}$ contains $\aleph_1$ points, it is evident that the set $S = \bigcup_{F \in \mathfrak{F}} F$ contains $\aleph_1 \cdot |\mathfrak{F}|$ points, and obviously $\aleph_1 \cdot |\mathfrak{F}| < 2^c$. Let $(C_\beta)_{\beta < \omega_1}$ be an indexing of $\mathfrak{B}$ (ω_1 is the first uncountable ordinal). By 2.1 C_0 contains 2^c P-points, so we can select $p_0 \in [C_0 \cap P(X^*)] - S$. Let γ be a countable ordinal, and for each $\beta < \gamma$, suppose we have selected $p_\beta \in [C_\beta \cap P(X^*)] - [S \cup \{p_\alpha\}_{\alpha < \beta}]$. As $|S \cup \{p_\beta\}_{\beta < \gamma}| < 2^c$, by 2.1 we can find $p_\gamma \in [C_\gamma \cup P(X^*)] - [S \cup \{p_\beta\}_{\beta < \gamma}]$. Let $D = (p_\beta)_{\beta < \omega_1}$. Then $\mathfrak{F} \cup \{D\}$ is a P-family of X^*, and $\mathscr{C} \cup \{\mathfrak{F} \cup \{D\}\}$ is a chain in $\mathscr{S}$ properly containing $\mathscr{C}$. This contradicts the maximality of $\mathscr{C}$, and so $|\mathfrak{F}| = 2^c$.

2.6 Lemma. *Let $(X_\alpha)_\alpha$ be a family of m-bounded subspaces of a space Y. Then $\bigcap_\alpha X_\alpha$ is a m-bounded subspace of Y.*

Proof. Trivial.

We can now prove Theorem 2.7 as stated in the introduction.

Proof of 2.7. As σ-compact spaces are Lindelöf and hence realcompact, by 2.5 we can find a P-family $\mathfrak{F}$ of X^* with $|\mathfrak{F}| = 2^c$. Let Σ denote the first ordinal of cardinality 2^c, and set $\mathfrak{F} = (F_\alpha)_{\alpha<\Sigma}$. For each $\alpha < \Sigma$, let S_α be the set of all points of X^* that are in the X^*-closure of some countable subset of F_α. Evidently S_α is $\aleph_0$-bounded. Since X is locally compact, σ-compact, and non-compact, it follows from theorem 14.27 of [3] that X^* is a compact F-space. If $\alpha \neq \beta$ then $S_\alpha \cap F_\beta = \emptyset$ as no point of F_β is a limit point of any countable subset of F_α (each point of F_β being a P-point of X^*). As $X^* - S_\alpha$ is $\aleph_0$-bounded by 2.3, it follows that X^*-closures of countable subsets of F_β are disjoint from S_α, and so $S_\alpha \cap S_\beta = \emptyset$. Thus $(S_\alpha)_{\alpha<\Sigma}$ is a family of 2^c pairwise disjoint $\aleph_0$-bounded subspaces of X^*.

Consider the space $H = X^* - \bigcup_{1<\alpha<\Sigma} S_\alpha$. Evidently $F_0 \subseteq H$ so H is dense in X^*. By 2.3 $X^* - S_\alpha$ is $\aleph_0$-bounded for each α, and hence by 2.6 H is $\aleph_0$-bounded. Thus $(S_\alpha)_{1<\alpha<\Sigma} \cup \{H\}$ is the desired decomposition of X^*.

2.7 Remark. M. E. Rudin has shown in [7] that there are non-P-points of N^* that are not limit points of any countable set of P-points of N^*. Hence if $X = N$ in the above theorem, it is evident that $\bigcup_{\alpha<\Sigma} S_\alpha \neq N^*$.

3. The remote points of βX.

3.1 Definition. A point $p \in \beta X$ is called a remote point of βX if p is not in the βX-closure of any discrete subspace of X.

Remote points were first defined by Fine and Gillman [2], who demonstrated, assuming the continuum hypothesis, the existence of a set of remote points in $\beta \boldsymbol{R}$ that is dense in $\boldsymbol{R}^*$ ($\boldsymbol{R}$ denotes the space of real numbers). Plank [6] has characterized the remote points of βX in the case where X is a separable metric space without isolated points. The following result is a combination of theorems 5.4 and 5.5 of [6].

3.2 Theorem. (Plank). *If X is a locally compact, σ-compact, non-compact metric space without isolated points, then the set of remote points of βX is precisely the set*

$$\bigcap_{Z \in \mathscr{Z}(X)} [X^* - (bd_X Z)^*],$$

where $\mathrm{bd}_X Z$ *denotes the topological boundary (in X) of Z. Assuming the continuum hypothesis, the remote points of βX form a dense subset of X^* of cardinality 2^c.*

We shall denote the set of remote points of βX by $T(X^*)$. Obviously $T(X^*) \subseteq X^*$ We wish to show that both $T(X^*)$ and $X^* - T(X^*)$ are $\aleph_0$-bounded spaces.

To do this, we need some information concerning the structure of locally compact, σ-compact spaces. The following result is well known; see, for example, page 241 of [1].

3.3 THEOREM. *A locally compact, σ-compact space can be written in the form $\bigcup_{n=0}^{\infty} V_n$, where for each n, V_n is open in X and $\mathrm{cl}_X V_n$ is compact and contained in V_{n+1}.*

The symbol "V_n" will henceforth be used with the above meaning.

The following lemma also appears in [9].

3.4 LEMMA. *Let $(A_n)_{n\in N}$ be a sequence of closed subsets of the locally compact, σ-compact space X. For each $n \in N$ define the non-negative integer k_n by*

$$k_n = \min\{j \in N : A_n \cap V_j \neq \emptyset\}.$$

If $\lim_{n\to\infty} k_n = \infty$, then $\bigcup_{n=0}^{\infty} A_n$ is closed in X.

PROOF. Let $p \in \mathrm{cl}_X(\bigcup_{n=0}^{\infty} A_n)$ and let U be an open subset of X containing p. There exists $i \in N$ such that $p \in V_i$; thus $(U \cap V_i) \cap (\bigcup_{n=0}^{\infty} A_n) \neq \emptyset$. As $\lim_{n\to\infty} k_n = \infty$, there exists $m \in N$ such that $n > m$ implies $A_n \cap V_i = \emptyset$. Thus $(U \cap V_i) \cap (\bigcup_{n=0}^{m} A_n) \neq \emptyset$, and so $U \cap (\bigcup_{n=0}^{m} A_n) \neq \emptyset$. As U was an arbitrary open set containing p, it follows that p belongs to the closed set $\bigcup_{n=0}^{m} A_n$, and so $p \in \bigcup_{n=0}^{\infty} A_n$. Thus $\bigcup_{n=0}^{\infty} A_n$ is closed.

We are now in a position to prove Theorem 3.5, which is stated in the introduction. Recall that X is locally compact if and only if $\beta X - X$ is compact.

PROOF OF 3.5. We first prove that $T(X^*)$ is $\aleph_0$-bounded. Let $D = (p_n)_{n\in N}$ be any countable subset of $T(X^*)$, and let $\mathscr{A}(p_n)$ be the z-ultrafilter on X associated with p_n (see [3], chapter 6). Put $\mathfrak{F} = \bigcap_{i=0}^{\infty} \mathscr{A}(p_i)$ and set $K = \bigcap_{F\in\mathfrak{F}} F^*$. Then $p_n \in F^*$ for each $n \in N$ and each $F \in \mathfrak{F}$, and so D is a subset of K, which is closed in the compact space X^* and hence is compact. It thus suffices to show that $K \subseteq T(X^*)$. Suppose that $q \notin T(X^*)$. By 3.2 there exists a closed nowhere dense subset Z of X such that $q \in Z^*$. Since p_n is a remote point, it follows that $Z \notin \mathscr{A}(p_n)$ for each $n \in N$, and so we can choose $A_n \in \mathscr{A}(p_n)$ such that $Z \cap A_n = \emptyset$. Since $\mathrm{cl}_X V_n$ is compact, we can without loss of generality assume that $A_n \subseteq X - V_n$ for each $n \in N$ (see (3.3)). It follows from 3.4 that $\bigcup_{i=0}^{\infty} A_i$ is closed in X. As closed subsets of metric spaces are zero-sets and as $A_n \subseteq \bigcup_{i=0}^{\infty} A_i$ for each $n \in N$, evidently $\bigcup_{i=0}^{\infty} A_i \in \mathfrak{F}$. Thus $K \subseteq (\bigcup_{i=0}^{\infty} A_i)^*$. But $Z \cap (\bigcup_{i=0}^{\infty} A_i) = \emptyset$ so by 6.5 (IV) of [3] we have $Z^* \cap (\bigcup_{i=0}^{\infty} A_i)^* = \emptyset$. Thus $q \notin K$ and so $K \subseteq T(X^*)$.

We next prove that $X^* - T(X^*)$ is $\aleph_0$-bounded. Let $D = (p_n)_{n \in N}$ be a countable subset of $X^* - T(X^*)$. Using 3.2 we can, for each $n \in N$, find a closed nowhere dense subset Z_n of X such that $p_n \in Z_n^*$, and without loss of generality we can assume that $Z_n \subseteq X - V_n$, since $\mathrm{cl}_X V_n$ is compact. It follows from 3.4 that $\bigcup_{i=0}^{\infty} Z_i$ is closed, and obviously $p_n \in (\bigcup_{i=0}^{\infty} Z_i)^*$ for each $n \in N$. Thus $\mathrm{cl}_{X^*} D \subseteq (\bigcup_{i=0}^{\infty} Z_i)^*$. Applying the Baire category theorem to the locally compact space X, we see that $\mathrm{int}_X(\bigcup_{i=0}^{\infty} Z_i) = \emptyset$; thus by 3.2 we have

$$(\bigcup_{i=0}^{\infty} Z_i)^* \subseteq X^* - T(X^*).$$

Hence $\mathrm{cl}_{X^*} D$ is a compact subset of $X^* - T(X^*)$.

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